**NUMERICAL ANALYSIS**

It is hard to describe our real-life problems with sophisticated mathematical formulas. Most of the cases complicated problems arise in a differential formation. Then finding the solution for this kind of equation is obvious. But most of the time it is hard to find out the exact solution of such kind of problems and sometimes there is no exact solution for the problems. In this scenario we need to estimate our best guess for the problem. There is various methodology to find out the most exact numerical value for the problems. Though we are not getting the exact value but we are very much close to the exact solution and sometimes the solution is nicely fitted. In 1614 after the creation of logarithms by John Napier, it has been an important aspect for numerical analysis.

**HISTORY OF NUMERICAL ANALYSIS:[4]**

A simple method for finding the root of a simple equation is found in Egyptian Rhind papyrus in 1650 BC. Next Eudoxus of Cnidus and Archimedes estimated the method for calculating areas volumes and length of any geometric figures. When Isaac Newton and Gottfried Leibniz developed the calculus, it has been a great tool to solve numerical problems. Calculus led to accurate mathematical models for physical reality, first in the physical sciences and eventually in the other sciences, engineering, medicine, and business. These mathematical models are usually too complicated to be solved explicitly, and the effort to obtain approximate, but highly useful, solutions gave a major impetus to numerical analysis. Newton created a number of numerical methods for solving a variety of problems, and his name is still attached to many generalizations of his original ideas. Particularly finding roots for general functions and finding a polynomial equation which fits best with the data that is polynomial interpolation. After Newton’s mathematician Swiss Leonhard Euler (1707–1783), the French Joseph-Louis Lagrange (1736–1813), and the German Carl Friedrich Gauss (1777–1855) contribute to the numerical analysis in 18th and 19th century.

After the creation of computer, numerical analysis become sophisticated. Critical and more complex analysis could be possible to do with the help of computer. Now it is impossible to solve numerical problems without computer.

**METHODS OF NUMERICAL ANALYSIS:[2]**

Depending on the problems we have to use various types of method for numerical analysis. Here we will consider the one variable equations.

**Numerical methods for ordinary differential equations:[1]**

For ordinary differential equations there are some method for finding approximate solution. They are,

**Euler method:**

This is a first order numerical procedure for solving ODEs with a given initial

value.

This method is named after Leonhard Euler. He described this method in his

book “Institutionum calculi integralis (published 1768–1870)”.

Euler method means that the local error is proportional to the square of the

step size, and the global error is proportional to the step size. The Euler

method often serves as the basis the construct more complex methods.

Consider that we want to find out the shape of an unknown curve which starts at a given point and satisfied a given differential equation. With This differential equation, slope of the tangent line of the curve can be determined.

The idea is that while the curve is initially unknown, its starting point, which we denote by , is known. Then, from the differential equation, the slope to the curve at can be computed, and so, the tangent line.

Take a small step along that tangent line up to a point . Along this small step, the slope does not change too much, so will be close to the curve. If we pretend that is still on the curve, the same reasoning as for the point above can be used. After several steps, a polygonal curve is computed. In general, this curve does not diverge too far from the original unknown curve, and the error between the two curves can be made small if the step size is small enough and the interval of computation is finite:

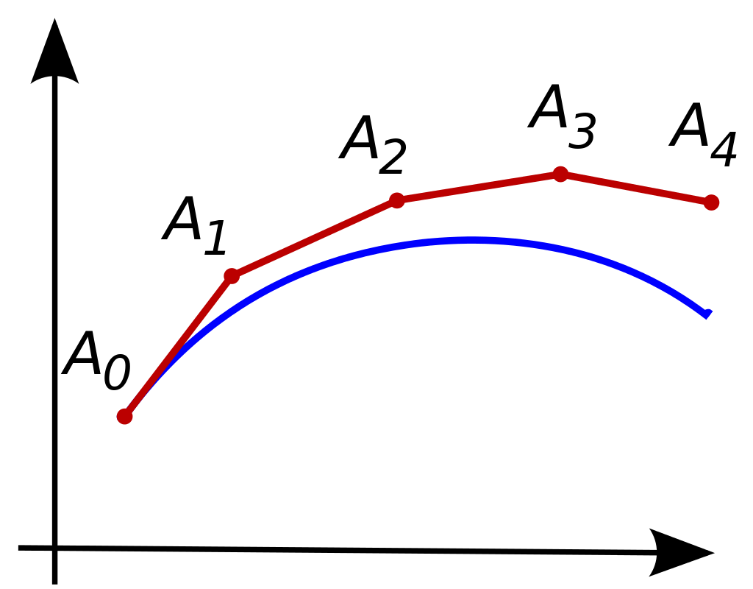
Choose a value h for the size of every step and set. Now, one step of the Euler method from is

The value of is an approximation of the solution to the ODE at time The Euler method is explicit, i.e. the solution is an explicit function of for .

While the Euler method integrates a first-order ODE, any ODE of order N can be represented as a system of first-order ODEs: to treat the equation

we introduce auxiliary variables and obtain the equivalent equation:

This is a first-order system in the variable and can be handled by Euler's method or, in fact, by any other scheme for first-order systems.



**Runge–Kutta methods:[1]**

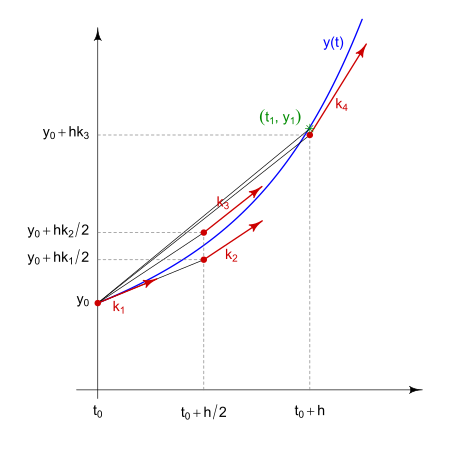
Considered the initial value problem

Where y is the unknown function of time t. We have to approximate this function. We have the rate of change of y and the initial condition. Now take a step size h>0 and let

For n = 0,1,2,3,… using

Here is the Runge-Kutta approximation of and the next value is determined by the dpresent value added with the weighted average of four increments where each increment is the product of the size of the interval , h, and an estimated slope specified by function f on the right-hand side of the differential equation.

* is the slope at the beginning of the interval, using y.
* is the slope at the midpoint of the interval , using y and .
* is again the slope at the midpoint, but now using y and .
* is the slope st the end of the interval , using y and .

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**Finite difference:[1]**

Finite difference method is written in the form The difference quotient is obtained by divided by . This method is first introduced by Brook Taylor in 1715.

There are three types in finite difference method, forward, backward and central finite difference method.

Forward difference is written in the form

Here h is the spacing between two points. This can be constant or variable.

Backward difference is written in the form

And the Central difference is given by

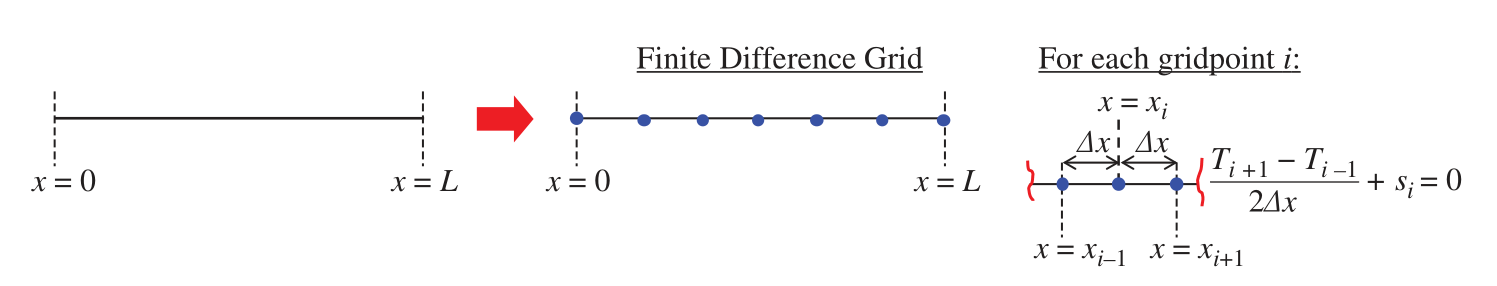
**FINITE ELEMENT METHOD [3]**

Before understanding finite element method let’s understand finite different method. Consider the differential equation

Where T(x) is unknown function which we need to find out and s(x) is a given function.

To find out the solution let us consider the central difference formula. Let divide the domain into finite interval with some grid points with equal space . For each location grid point , we use the two nonboring points , to write the following approximation of the derivative at ,

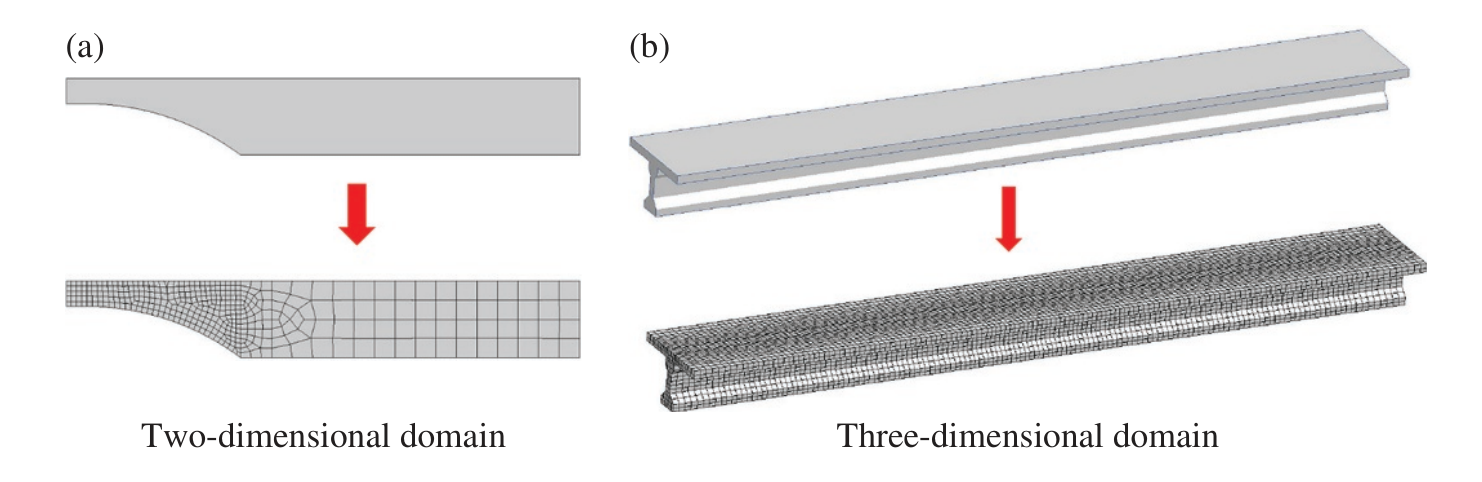
Now putting this value in the equation, we get



**figure**

Where . For each grid point we will get a system of equation which will allow us to find out the unknown function T(x).

In Finite Element method first, we need to discretize or subdivide the physical geometric domain into smaller subdomains which are called elements. Now the function governs each element are called field functions. We have to setup the field function, usually these functions are polynomial which contain some parameter which we need to obtain.



**HISTORY OF FINITE ELEMENT METHOD [3]**

It is not clear who introduced finite element method first. But it is accepted that this method is developed from aerospace engineering research in the 1950s. Professors Jon Turner (in the United States) and John Argyris (in Europe) are the two initial contributors of finite element method. In 1953, Turner came up with the idea of pursuing improved models of aircraft wings through representation of the skin with triangular elements, having constant stress. After some years Turner el al., Clough realized the importance of direct stiffness method. Next time they named it finite element method. In 1960 he published a paper mentioning the name. Major early contributions on finite element analysis also came from Professor Olgierd Zienkiewicz at the University of Swansea, Wales, and his collaborators. After the mathematical rigorous verification of the validity of the FEM, the method became vastly more popular, primarily thanks to research conducted at UC Berkeley by Taylor and Wilson.

**BASIC ALGORITHM OF FINDING THE FINITE ELEMENT SOLUTION: [3]**

\*First, we need to define the Governing differential equation for the physical problem with the boundary condition which is known as **strong form**.

Where E(x) → Young’s modulus, A(x) → cross-sectional area of a metal road where force is applied to contract or stress the metal road.

\*Then we have to consider a more convenient equation as a solution which is closer to the main solution which is called the weak form.

**\***Next the finite element approximation is used.

Here [N] is a matrix with known function of x which are chosen and {U} are unknown parameters which have to determine.

\*If we plug the approximation in the weak form, we eventually obtain an algebraic system of linear equations which we can solve for {U}.

**VARIOUS FINITE ELEMENT METHODS:[5]**

**Applied element method(AEM):[1]**

In numerical analysis, applied element method is used to predict continuum and discrete behavior of structures. In AEM, the structure is divided virtually and modeled as an assemblage of relatively small elements. The elements are then connected through a set of normal and shear springs located at contact points distributed along the element faces. Normal and shear springs are responsible for the transfer of normal and shear stresses from one element to the next. In AEM the elements are connected by a series of non-linear springs representing the material behavior. Specially three types of springs are used. They are Matrix Springs, Reinforcing Bar Springs and Contact Sprigs. The spring stiffness in a 2D model can be calculated from the following equations,

Where d is the distance between springs, T is the thickness of the element, a is the length of the representative area, E is the Young's modulus, and G is the shear modulus of the material. The above equation's indicate that each spring represents the stiffness of an area (T·d) within the length of the studied material.

This method is being used in, Structural vulnerability assessment like Progressive collapse, Blast analysis, Impact analysis, Seismic analysis. This method is also use in Forensic engineering, Performance based design, Demolition analysis, Glass performance analysis, Visual effects.

**Mixed finite element method:[1]**

In Mixed finite element method, extra fields to be solved are introduced during the posing a partial differential equation problem. Somewhat related is the hybrid finite element method. The extra fields are constrained by using Lagrange multiplier fields. To be distinguished from the mixed finite element method, usual finite element methods that do not introduce such extra fields are also called irreducible or primal finite element methods. The mixed finite element method is efficient for some problems that would be numerically ill-posed if discretized by using the irreducible finite element method; one example of such problems is to compute the stress and strain fields in an almost incompressible elastic body.

**Finite element limit analysis:[1]**

This method uses optimization techniques to directly compute the upper or lower bound plastic collapse load (or limit load) for a mechanical system rather than time stepping to a collapse load, as might be undertaken with conventional non-linear finite element techniques. The problem may be formulated in either a kinematic or equilibrium form. The technique has been used most significantly in the field of soil mechanics for the determination of collapse loads for geotechnical problems (e.g. slope stability analysis). An alternative technique which may be used to undertake similar direct plastic collapse computations using optimization is Discontinuity layout optimization.

**Galerkin method:**

In the area of numerical analysis, Galerkin Methods is used to convert a continuous operator problem to a discrete problem**.** Most of the cases the problems like differential equation commonly in a weak formulation. There are some verities in Galerkin method.[1]

Consider the boundary value problem

Now consider the approximate trial solution as

Which satisfy the boundary condition and in a simple form. Here is a parameter. It is clear that  . Now we have to find out the value of for which we can get the best result. Now the main equation can be written as

If we put the approximate value in this equation then we will not get zero. So, putting the approximate solution we can write

This function is called the residual function. It is clear that if the approximate solution coincides with the exact solution, then for all x in the domain.

A number of techniques based on the use of residual functions are available for determining the unknown parameters in the trial solutions.

One such method is called the collection method which involves forcing the approximate solution to be exact at n points where n is the number of unknown parameters. This implies that

Which is a set of n equations to be solved for the parameters.

**Galerkin’s Weighted Residual Method:**

Galerkin’s method involves the concept of weighted residuals. In this method we determine the n unknown parameters by selecting n weighting functions and requiring that each of the n integrals

In Galerkin’s method we use as weighting functions those terms in the trial solution which are multiplied by unknown parameters. Considering the trial function, we can take

So, the residual function can be written as

Now the Galerkin procedure requires to be such that

So, we get

After doing the integration we get

Then our approximation solution is

Now the exact solution to this boundary value problem is known and the exact solution is

The graph of both trial and exact solution with the residual function are shown in the graph.

Gaph graph graph graph graph ……………………………..

The formal statement of the Galerkin approximation procedure at least for linear second order one dimensional boundary value problems of the general form:

With given boundary conditions as x = a and x = b

Approximate trial solutions are used which are of the form

Here

* is chosen so that it satisfies the boundary conditions of the problem.
* The functions must each satisfy the corresponding homogeneous form of the boundary conditions. These functions are called coordinate functions.

**HISTORY OF GALERKIN METHODS:[1]**

This method is named after Borish Galerkin. Hencky and Duncan explained this method to the Western reader. The convergence of Galerkin method was studied by MIkhlin andj Leipholz . Its coincidence with Fourier method was illustrated by Elishakoff et al. Singer showed the equivalence to Ritz’s method for conservative problems. Gander and Wanner showed how Ritz and Galerkin methods led to the modern finite element method. One hundred years of method's development was discussed by Repin.

**PREVIOUS WORK ON GALERKIN METHOD:**

**VARIOUS TYPEPS OF GALERKIN METHODS:**

**MODIFIED GALERKIN METHOD:[5]**

Consider the one-dimensional Poisson equation

With one non-derivative boundary conditions say and one derivative boundary condition say at . The function is given.

Let’s take a trial solution such that

Then the weighted residual equation is

For where n is the number of unknown parameters.

Now integrating the first term by parts we get

**RITZ-GALERKIN METHOD:[1]**

This method is named after Walter Ritz. It is also called Rayleigh-Riz method and the Ritz-Galerkin method. The Ritz-Galerkin method is a direct method for finding an approximated solution for boundary value problems. First a wave function is assumed which satisfies the boundary condition of the problem. The function contains some adjustable parameter which is used to find out the satisfied solution.

**DISCONTINUOUS GALERKIN METHOD:**

**HISTORY OF DG METHOD:**

Discontinuous Galerkin methods were first proposed and analyzed in the early 1970s as a technique to numerically solve partial differential equations. In 1973 Reed and Hill introduced a DG method to solve the hyperbolic neutron transport equation. The origin of the DG method for elliptic problems cannot be traced back to a single publication as features such as jump penalization in the modern sense were developed gradually. However, among the early influential contributors were Babuška, J.-L. Lions, Joachim Nitsche and Miloš Zlámal. DG methods for elliptic problems were already developed in a paper by Garth Baker in the setting of 4th order equations in 1977. A more complete account of the historical development and an introduction to DG methods for elliptic problems is given in a publication by Arnold, Brezzi, Cockburn and Marini. A number of research directions and challenges on DG methods are collected in the proceedings volume edited by Cockburn, Karniadakis and Shu.

**WHY DG METHOD:[6]**

* The actual order of accuracy of DG methods only depends on the exact solution; DG methods of arbitrarily high formal order of accuracy can be obtained by suitably choosing the approximating polynomials.
* DG methods are highly parallelizable. Since the elements are discontinuous, the mass matrisx is block diagonal and since the size of the blocks is equal to the number of degrees of freedom inside the corresponding elements, the blocks can be inverted by hand onc3 and for all.
* DG methods are very well suited to handling complicated geometries and require an extremely simple treatment of the boundary conditions in order to achieve uniform high-order accuracy.
* DG methods can easily handle adaptivity strategies since refinement or unretirement, of the grid can be achieved without taking into account the continuity restrictions typical of conforming finite element, method. Moreover, the degree of the approximating polynomial can be easily changed from one element to the other. Adaptivity is of particular importance in hyperbolic problems given the complexity of the structurer of the discontinuities.

**MAIN DISCONTINUIOUS GALERKIN METHOD:[6]**

In 1973 mathematician Reed and Hill introduce this method for solving neutron transport equation.

The neutron transport equation is,

Here, is a real number and **a** is a constant vector.

Now we multiply this equation with a test function and integrate over an arbitrary subset of . By integrating by parts, we get,

Here is the outward unit normal of and

Now constructing a triangulation of , and taking approximate solution to be polynomial of degree at most on each element of the triangulation. The approximate solution is then determined as the unique solution of the following weak formula:

Here, is the polynomials space of degree at most on the element and is the numerical flux given by

Here the value Is nothing but the value of upstream the characteristic direction . As a consequence, the degrees of freedom of the approximate solution in the element can be computed in terms of the values of upstream the characteristics hitting . So, the approximate solution can be computed element by element when the elements are suitably ordered according to the characteristic direction **.**

**ALGORITHM OF DG METHOD:**

**EARLY APPLILCATIONS OF DG METHOD:[6]**

With the application of the DG method for simulating the neutron transport problem

**AN EXAMPLE ON DG METHODS:**

**CENTRAL DIFFERENCE FORMULA**

There are a lot of numerical method to solve differential equation. Now we are going to define central difference method. Consider the following differential equation,

Where are known function, which are continuous in the interval [a, b]. are single and double derivative of y. Let N>0 be an integer and divide the interval [a, b] into (N+1) equal subinterval. The end points of each interval are the mesh points. Let the mesh points , where . The cause of Taking the subinterval N+1 so that a Linear system can be formed with matrix can be formed.

Now at the mesh points the approximate differential equation can be written as

Now by Taylor polynomial expansion we can get,

For some , and

For some .

Now adding (4) and (5) we get

Now using intermediate value Theorem, we can write

For some ,

Now we get the value of from (6)

Now similarly subtracting (4) and (5) we get the value of as,

For some in .

Now putting this value in (1) we get,

Let , then with the truncation error or order results by using this equation together with the boundary conditions to define the system of linear equations

And

for each i = 1, 2, . . ., N.

From (11) we can represent the equation as

For i=1,2, 3, … N, the resulting system of (12)j can be written in the tridiagonal matrix form,

Where,

By solving the matrix, we can get the approximate values in the mesh points.

This is for linear differential equation. Now consider the partial differential equation.

On , with for , where S is the boundary of R. If f and g are continuous on their domains, then there is a unique solution to this equation.

For this two-dimensional problem we have to choose integer N and M greater than zero. Let’s divide the interval [a, b] into N subinterval and [c, d] into M subinterval. Then the mesh points can be written as

Where .

And are the grid lines and the intersection points of the grid lines are the mesh points for this 2d system. So similarly like the linear problem by Tailor series we can write the centered-difference formula for x variable about ,

Here , also for y variable about , after Tailor series expansion we get the centered-difference formula

Where

Putting this values in (13) we get

for each i = 1, 2, . . . , n − 1 and j = 1, 2, . . . , m − 1. The boundary conditions are

Now the finite difference method is

for each i = 1, 2, . . ., n − 1 and j = 1, 2, . . ., m − 1, and

where approximates . This method has local truncation error of order

**AN EXAMPLE ON CENTRAL DIFFERENCE FORMULA:**

**COMPARISN BETWEEN THE TWO METHODS MATH:**

**CONCLUSION:**

**REFERENCE**

**1.Wikipedia**

**2.Numerical analysis by berden**

**3.** **Fundamentals of Finite Element Analysis (Ioannis Koutromanos)**

**4.Britanica**

**5.The finite Element Method (P.E. Lewis and J.P. Ward)**

**6. The development of Discontinuous Galerkin Methods (Bernardo Cockburn, George E. Karniadakis, and Chi-Wang shu)**